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TD 3-Sheaves on topological spaces, ringed spaces

We write $\mathrm{Sh}(X)$, $\mathrm{Ab}(X)$ for the categories of sheaves of sets, resp. abelian sheaves on a topological space X. I will simply write equality instead of "canonical isomorphism". If $F \in \mathrm{Sh}(X)$ and $x \in X$, F_x is the stalk of F at x, and for $s \in F(U)$ with $x \in U$, s_x is the image of s in F_x.

A **ringed space** $X = (X, O_X)$ is a topological space X together with a sheaf of rings O_X on it, and a morphism of ringed spaces $f = (f, f^\sharp) : X \to Y$ consists of a map of topological spaces $f : X \to Y$ and a map of sheaves of rings $f^\sharp : O_Y \to f_*O_X$. We write $\mathrm{Mod}(X)$ for the category of O_X-modules (i.e. abelian sheaves F such that $F(U)$ is an $O_X(U)$-module in a way compatible with restrictions) on a ringed space (X, O_X). If $f : X \to Y$ is a morphism of ringed spaces and $F \in \mathrm{Mod}(Y)$, we define $f^*(F) \in \mathrm{Mod}(X)$ by

$$f^*F = O_X \otimes_{f^{-1}(O_Y)} f^{-1}(F),$$

the map $f^{-1}(O_Y) \to O_X$ being induced (by adjunction) by $f^\sharp$ and the tensor product sheaf being the sheafification of the obvious presheaf.

X **is always a topological space below, sometimes-always mentioned-a ringed space**.

0.1 A few concrete examples

1. What is the sheafification of the presheaf on $\mathbf{R}$ sending U to the set of continuous bounded maps $f : U \to \mathbf{R}$?
2. a) Let F be the sheaf of holomorphic functions on $\mathbf{C}$, and let $f : F \to F$ be the map induced by $\frac{d}{dz}$. Is the presheaf $U \to \mathrm{coker}(f(U) : F(U) \to F(U))$ a sheaf ? What is its sheafification ?

   b) Let $X = \mathbf{C}^*$ and consider the exponential as a map between the sheaf F of continuous $\mathbf{C}$-valued functions on X and the sheaf G of invertible continuous maps on X. Is the presheaf $U \to \mathrm{Im}(\exp(U) : F(U) \to G(U))$ a sheaf ? What is its sheafification ?
3. Let $X = \mathbf{C}^*$ and consider the map $f : X \to X$ sending z to z^2. Is the direct image $f_*\mathbf{C}$ of the constant sheaf $\mathbf{C}$ a constant sheaf ?
4. Let $(A_x)_{x\in X}$ be arbitrary sets, indexed by a topological space X. Prove that there is a natural sheaf F on X such that for all open subsets U of X we have $F(U) = \prod_{x\in U} A_x$. If A_x are abelian sheaves and we replace direct product by direct sum, is the conclusion valid ? Is A_x the stalk of F at $x \in X$?

0.2 Espace étalé

Let F be a sheaf on X. Let $\tilde{F} = \coprod_{x\in X} F_x$ and $\pi : \tilde{F} \to X$ the natural map. If $s \in F(U)$, define a map $g_s : U \to \tilde{F}$, $g_s(x) = s_x \in F_x$, and consider the topology on $\tilde{F}$ having as open sets the $g_s(U)$.

1. Prove that the topology induced on F_x is the discrete topology, and that g_s is a homeomorphism from U onto its image.
2. Prove that $F(U)$ is identified with the set of continuous functions $s : U \to \tilde{F}$ such that $\pi \circ s = \mathrm{id}_U$.

0.3 Everything is seen by stalks

Let $f, g : F \to G$ be maps of presheaves on a topological space X.

1. Prove that if F is a sheaf, then for any open $U \subset X$ the natural map $F(U) \to \prod_{x\in U} F_x$ is injective.
2. If G is a sheaf and $f_x = g_x : F_x \to G_x$ for all $x \in X$, then $f = g$.
3. If F is a sheaf, prove that $f(U) : F(U) \to G(U)$ is injective for all open subsets U if and only if $f_x : F_x \to G_x$ is injective for all $x \in X$ (in which case we say that f is injective). The same happens with injective replaced by bijective, if we assume moreover that G is a sheaf.

4. Let X be an open subset of $\mathbf{C}$ and consider the sheaf O_X of holomorphic functions on (open subsets of) X. Prove that the map $D : O_X \to O_X$, $D(f) = f'$ is a surjective map of sheaves (i.e. it induces surjective maps on all stalks), and give an example where the induced map on sections over X is not surjective.
5. Consider an exact sequence of sheaves $0 \to E \to F \to G \to 0$ in $\mathrm{Ab}(X)$ (i.e. the map $F \to G$ is surjective as a map of sheaves, and its kernel-computed in the naive way-is E). Prove that for all open subsets U of X we have an exact sequence $0 \to E(U) \to F(U) \to G(U)$. Give examples where this sequence is not exact on the right.

0.4 New sheaves out of old ones

Let X be a topological space.

1. (products/direct sums) Let $(F_i)_{i\in I}$ be a family of abelian sheaves on X.
   a) Prove that $U \to \prod_{i\in I} F_i(U)$ is an abelian sheaf on X and has the expected universal property.
   b) Prove that $U \to \oplus_{i\in I} F_i(U)$ is not always a sheaf on X, but its sheafification, denoted $\oplus_{i\in I} F_i$, has the expected universal property. Describe the stalks of $\oplus_{i\in I} F_i$. Also, prove that if U is a quasi-compact open subset of X, the natural map $\oplus_i F_i(U) \to (\oplus_i F_i)(U)$ is bijective.
2. If you know what inductive/projective systems and their limits are, extend the previous discussion to them (be careful though that the last part in b) above is no longer true for arbitrary colimits).
3. (glueing sheaves) Let $(U_i)_{i\in I}$ be an open covering of X. Let $F_i \in \mathrm{Sh}(U_i)$ be sheaves, together with isomorphisms $\varphi_{ij} : F_i|_{U_i\cap U_j} \simeq F_j|_{U_i\cap U_j}$ such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_i \cap U_j \cap U_k$ for all $i, j, k \in I$. Prove that there is a sheaf F on X together with isomorphisms $\varphi_i : F|_{U_i} \simeq F_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on $U_i \cap U_j$ for all i, j. Moreover, F and $(\varphi_i)_i$ are uniquely determined (up to unique isomorphism).
4. (tensor product) Let (X, O_X) be a ringed space. Define, for $F, G \in \mathrm{Mod}(X)$, $F \otimes_{O_X} G \in \mathrm{Mod}(X)$ as the sheafification of $U \to F(U) \otimes_{O_X(U)} G(U)$.
   a) Prove that $(F \otimes_{O_X} G)_x = F_x \otimes_{O_{X,x}} G_x$ for all $x \in X$.
   b) Let $f : X \to Y$ be a morphism of ringed spaces. See the introduction for the functor f^*.
   i) Prove that for $F, G \in \mathrm{Mod}(Y)$ we have $f^*(F \otimes_{O_Y} G) = f^*F \otimes_{O_X} f^*G$.
   ii) Let $F \in \mathrm{Mod}(X)$ and let $G \in \mathrm{Mod}(Y)$, with G locally free of finite rank (i.e. each $y \in Y$ has an open neighborhood U on which $G|_U$ is isomorphic to $O_U^{n_U}$ for some integer n_U). Prove the **projection formula**
   $$f_*(F \otimes_{O_X} f^*(G)) = f_*(F) \otimes_{O_Y} G.$$
5. (Hom sheaf) a) Let $F, G \in \mathrm{Ab}(X)$. Prove that $U \to \mathrm{Hom}_{\mathrm{Ab}(U)}(F|_U, G|_U)$ is again an abelian sheaf, called $\mathit{Hom}(F, G)$. What happens if we try to consider instead $U \to \mathrm{Hom}_{\mathrm{Ab}}(F(U), G(U))$?
   b) Suppose now that $X = (X, O_X)$ is a ringed space. We have an obvious variant of a), that we still call $\mathit{Hom}(F, G)$, sending U to $\mathrm{Hom}_{O_U}(F|_U, G|_U)$.
   i) Prove that if $F \in \mathrm{Mod}(X)$ is finitely presented (i.e. any point $x \in X$ has an open neighborhood U in X for which there is an exact sequence of O_U-modules $O_U^m \to O_U^n \to 0$ for some integers m, n depending on U), then for all $G \in \mathrm{Mod}(X)$ and $x \in X$ we have $\mathit{Hom}(F, G)_x = \mathrm{Hom}_{O_{X,x}}(F_x, G_x)$.
   ii) Prove that for any **flat** morphism of ringed spaces $f : X \to Y$ (flatness means that the map of rings $O_{Y,f(x)} \to O_{X,x}$ is flat for all $x \in X$) and any $F, G \in \mathrm{Mod}(X)$, with F finitely presented
   $$f^*\mathit{Hom}(F, G) = \mathit{Hom}_{O_X}(f^*F, f^*G).$$

0.5 Restriction to open and closed subspaces

Let $j : U \to X$, resp. $i : Z \to X$ be the inclusion of an open, resp. closed subset of a topological space X.

1. Describe concretely $j^{-1}F$ (often written $F|_U$) and prove that $j^{-1}j_*(F) = F$ for $F \in \mathrm{Ab}(U)$.
2. Let $j_! : \mathrm{Ab}(U) \to \mathrm{Ab}(X)$ be the **extension by zero** functor, so $j_!(F)$ is the sheafification of the presheaf sending V to $\{0\}$ when V is not contained in U, and to $F(V)$ otherwise.
   a) Prove that
   $$\mathrm{Hom}_{\mathrm{Ab}(X)}(j_!(F), G) = \mathrm{Hom}_{\mathrm{Ab}(U)}(F, j^{-1}(G)).$$

b) Prove that $(j_!F)_x$ is 0 when $x \notin U$ and F_x otherwise. Deduce that $j_!$ is an exact functor, identifying $\mathrm{Ab}(U)$ with the category of abelian sheaves on X whose stalks vanish outside U (start by checking that $j^{-1}j_!(F) = F$ for $F \in \mathrm{Ab}(U)$).

3. Prove that $(i_*F)_x$ is 0 when $x \notin Z$ and F_x otherwise. Deduce that i_* is an exact functor, identifying $\mathrm{Ab}(Z)$ with the category of abelian sheaves on X whose stalks vanish outside Z (start by checking that that $i^{-1}i_*(F) = F$ for $F \in \mathrm{Ab}(Z)$).
4. Suppose that $Z = X \setminus U$. Prove that for any $F \in \mathrm{Ab}(X)$ one has a canonical exact sequence

$$0 \to j_!j^{-1}(F) \to F \to i_*i^{-1}(F) \to 0.$$

5. a) Prove that any $F \in \mathrm{Ab}(X)$ has a largest abelian subsheaf $\mathcal{H}_Z(F)$ whose support is contained in Z. **Hint** : the sections of $\mathcal{H}_Z(F)$ are those $s \in F(U)$ whose support is contained in $Z \cap U$.
b) Define $i^! : \mathrm{Ab}(X) \to \mathrm{Ab}(Z)$ by $i^!(F) = i^{-1}\mathcal{H}_Z(F)$. Prove that $i^!$ is right adjoint to i_*, i.e.

$$\mathrm{Hom}_{\mathrm{Ab}(X)}(i_*G, F) = \mathrm{Hom}_{\mathrm{Ab}(Z)}(G, i^!F).$$

0.6 Flasque sheaves

An abelian sheaf F on a topological space X is called **flasque** if the restriction map $F(X) \to F(U)$ is surjective for all open subsets U of X.

1. Check that F is flasque iff $F(V) \to F(U)$ is surjective for all open subsets $U \subset V$ of X.
2. Prove that being flasque is stable under restriction to an open subset and under direct image by a continuous map.
3. Prove that a constant sheaf on an irreducible space is flasque.
4. a) Prove that if $0 \to E \to F \to G \to 0$ is an exact sequence in $\mathrm{Ab}(X)$, with E flasque, then $0 \to E(U) \to F(U) \to G(U) \to 0$ is exact for all open subsets U of X. Moreover, if E and F are flasque, then so is G.
b) Consider a long exact sequence of flasque sheaves $0 \to E \to F^0 \to F^1 \to F^2 \to ...$ in $\mathrm{Ab}(X)$. Prove that for all open subsets U of X we have a long exact sequence $0 \to E(U) \to F^0(U) \to F^1(U) \to ....$

0.7 Godement resolution, flasque sheaves, cohomology

Let X be a topological space. If $F \in \mathrm{Ab}(X)$, set $G(F)(U) = \prod_{x \in U} F_x$ for $U \subset X$ open.

1. Prove that $G(F)$ is a flasque sheaf, and that the natural map $F \to G(F)$ is injective.
2. Define a sequence of sheaves $Q^n(F), G^n(F)$ for $n \geq 0$, with natural injective maps $Q^n(F) \to G^n(F)$ as follows : $G^0(F) = G(F)$, $Q^0(F) = F$, and for $n \geq 1$ set

$$Q^n(F) = \mathrm{coker}(Q^{n-1}(F) \to G^{n-1}(F)), \quad G_n(F) = G(Q^n(F)),$$

the map $Q^n(F) \to G^n(F)$ being the natural one.
a) Prove that $F \to G^n(F)$ are exact functors for $n \geq 0$, and that $G^n(F)$ is a flasque sheaf for all n.
b) Prove that there is a long exact sequence of sheaves, called the **Godement resolution of** F

$$0 \to F \to G^0(F) \to G^1(F) \to G^2(F) \to ...$$

3. If U is an open subset of X and $F \in \mathrm{Ab}(X)$, we define the cohomology groups of F over U $H^n(U, F)$ as the cohomology groups of the induced complex $0 \to G^0(F) \to G^1(F) \to ...$ i.e. ($G^n(F) = 0$ for $n < 0$)

$$H^n(U, F) = \frac{\ker(G^n(F) \to G^{n+1}(F))}{\mathrm{Im}(G^{n-1}(F) \to G^n(F))}.$$

a) Check that $H^0(U, F) = F(U)$ and that if F if flasque, then $H^n(U, F) = 0$ for $n \geq 1$.
b) Prove that if $0 \to E \to F \to G \to 0$ is an exact sequence in $\mathrm{Ab}(X)$, then for any open subset U of X we obtain a long exact sequence

$$0 \to E(U) \to F(U) \to G(U) \to H^1(U, E) \to H^1(U, F) \to H^1(U, G) \to H^2(U, E) \to H^2(U, F) \to ...$$